Understanding the Higgs mass in string theory

Steve Abel *(IPPP)* 12/03/21

Mainly based on forthcoming work with Keith Dienes arXiv:2103.xxxxx and related to …

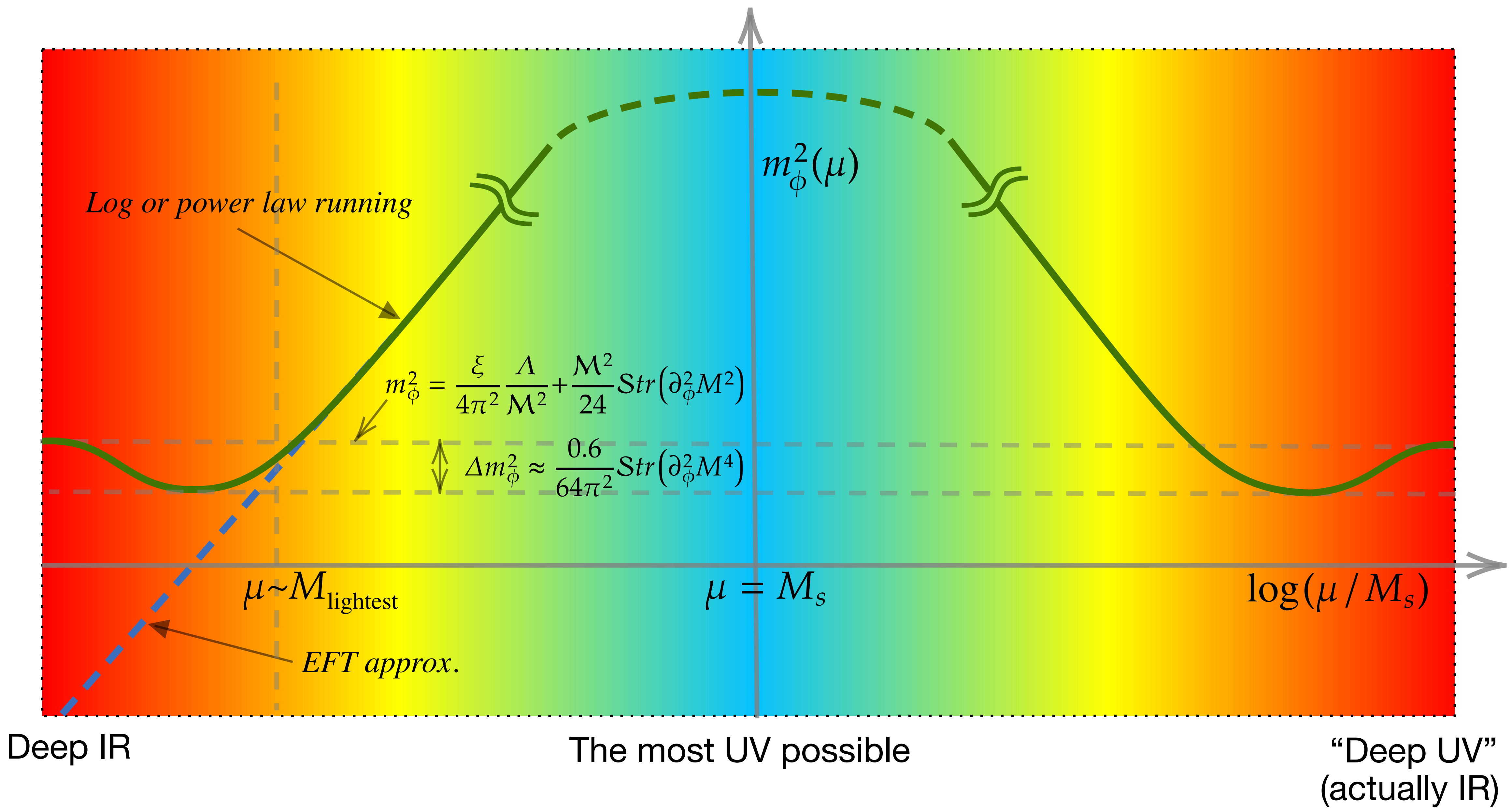
- w/ Dienes+Mavroudi *Phys.Rev. D* 91, (2015) 126014, arXiv:1502.03087
- SAA JHEP 1611 (2016) 085, arXiv:1609.01311
- Aaronson, SAA, Mavroudi, *Phys. Rev. D* 95, (2016) 106001, arXiv:1612.05742
- w/ Stewart, *Phys.Rev.D* 96 (2017) 10, 106013 arXiv:1701.06629
- w/ Dienes+Mavroudi *Phys.Rev.D* 97 (2018) 12, 126017 arXiv: 1712.06894

Take away picture of Higgs mass in strings:

Q (2017): “Suppose nature is a closed string theory. It is finite due to symmetries (modular invariance) that must be valid even today. Surely this must tell us something about the Higgs mass?”

Take away picture of Higgs mass in strings:

A (2021): “The Higgs mass begins at a UV value we can calculate, has RG running, maybe GUT breaking, EW and QCD phase transition, yada yada yada. But then it must eventually wind up at the exact same value in the IR. And everything is finite. Like this…”



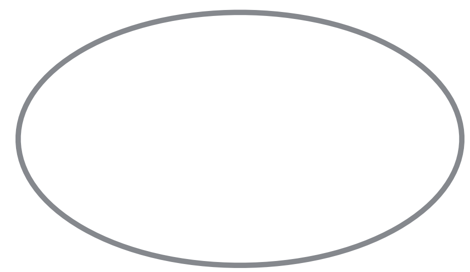
Layout

- Background: naturalness in effective field theory — key questions for the UV complete theory
- Modular invariance — the ultimate UV/IR mixer
- Modular constraints on the Higgs mass: relation between the Higgs mass and the Cosmological constant
- Regulating the Higgs mass: see how it runs!

1. Higgs naturalness in field theory: key questions

Or: why field theory is a totally unsuitable formalism for even thinking about this!

Let's look at one-loop cosmological constant (a.k.a. Coleman Weinberg potential). Loop of massive propagators as follows:



For later reference this can be written in a "stringy way" using a Schwinger worldline parameter, t :

$$\Lambda = \sum_i \int \frac{d^4k}{(2\pi)^4}(-1)^F \log(k^2+M_i^2) = \sum_i \int \frac{d^4k}{(2\pi)^4} \int \frac{dt}{t}(-1)^F e^{-t(k^2+M_i^2)}$$

$$= \sum_i \int_{M_{UV}^{-2}}^{\mu_{UV}^{-2}} \frac{dt}{t^3}(-1)^F e^{-t\,M_i^2}$$

Can identity a partition function in this way:

$$\boxed{\mathcal{Z}(t) = \mathrm{Str}\left(t^{-2}e^{-tM^2}\right)}$$

Performing the integral gives Coleman-Weinberg potential:

$$\Lambda = \frac{M_{UV}^2}{32\pi^2} \underset{EFT}{\mathrm{Str}} M^2 + \frac{1}{64\pi^2} \underset{EFT}{\mathrm{Str}} M^4 \log\left(const \times \frac{M^2}{M_{UV}^2}\right)$$

From which we can infer a running Higgs mass-squared:

$$m_\phi^2 = m_\phi^2(M_{UV}) + \frac{M_{UV}^2}{32\pi^2} \underset{EFT}{\mathrm{Str}} \partial_\phi^2 M^2 + \frac{1}{64\pi^2} \underset{EFT}{\mathrm{Str}} \partial_\phi^2 M^4 \log\left(const \times \frac{M^2}{\mu^2}\right)$$

Key Questions:

$$m_\phi^2 = m_\phi^2(M_{UV}) + \frac{M_{UV}^2}{32\pi^2}\underset{EFT}{\mathrm{Str}}\,\partial_\phi^2 M^2 + \frac{1}{64\pi^2}\underset{EFT}{\mathrm{Str}}\partial_\phi^2 M^4 \log\left(const \times \frac{M^2}{\mu^2}\right)$$

$$\boxed{\underset{EFT}{\mathrm{Str}}\,\partial_\phi^2 M^2 \ll 32\pi^2 \frac{m_W^2}{M_{UV}^2}}$$

- Is there any meaning at all to the oft-considered Veltman condition? Note that the above supertrace is over the effective theory only. But why should the whole UV complete theory care about just that?!
- What determines if a field is light enough to be called massless? e.g. GUT states do not contribute in e.g. the effective SM even though their mass is much less than the cut-off?!
- What is the right UV regulator? — e.g. dimensional regularisation doesn't give a leading quadratically UV sensitive piece.
- Where does the Higgs mass run *to* in the IR? Where and how do we stop it?
- Note the mass is both UV hypersensitive and IR divergent if there are massless states (the only operator that is). How can this operator be regulated at both ends at the same time?

Key Questions:

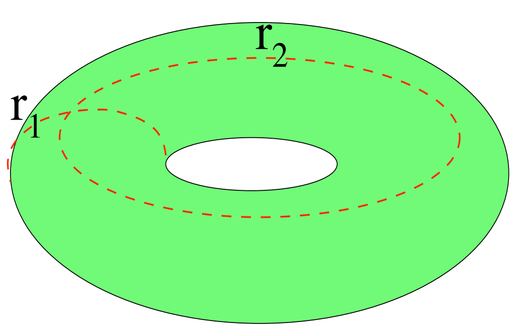
- ***In short**: the problem is that we are trying to guess how a parameter might behave because of the UV completion, when we don't know the UV completion.*
- *Non-SUSY strings are an interesting laboratory to address these questions properly*
- *Plus if you confine to SUSY strings you are frankly blind to the beauties of number theory*
- *(Note the world is not because it is not supersymmetric)*
- ***Warning**: in this talk I do not attempt to construct a non-supersymmetric model. I just want to draw general conclusions about the properties the Higgs mass must have (even today) due to the theory's finiteness.*

2. Modular invariance

Or: the ultimate UV/IR mixer

Let's understand how string theory gets to be finite: Revisit the cosmological constant but now in string theory

Closed string theory maps out a torus:

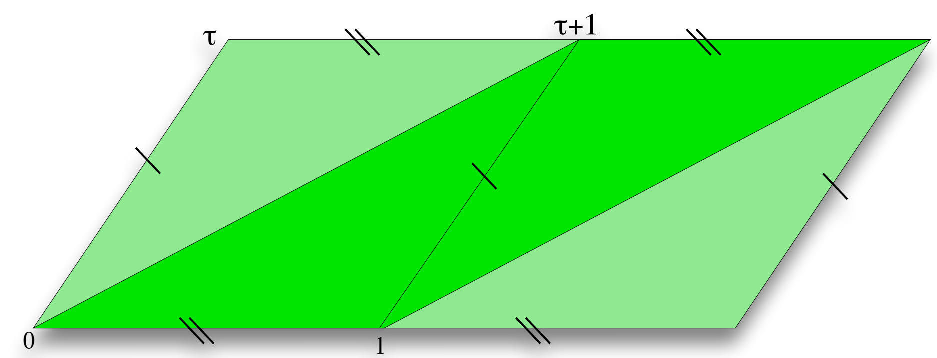


Let's understand how string theory gets to be finite: Revisit the cosmological constant but now in string theory

Closed string theory maps out a torus:

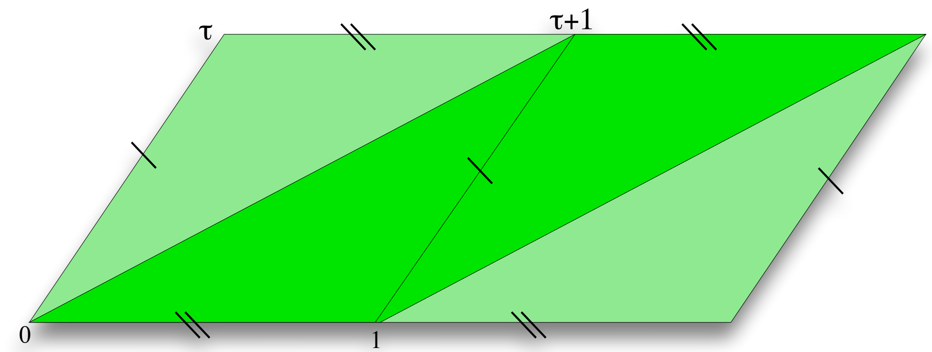
can be mapped to complex plane, τ

but theory invariant under modular transformations:



$\tau \to \tau + 1$ redefines torus :

$\tau \to -1/\tau$ swops σ_1 and σ_2 and just reorients torus



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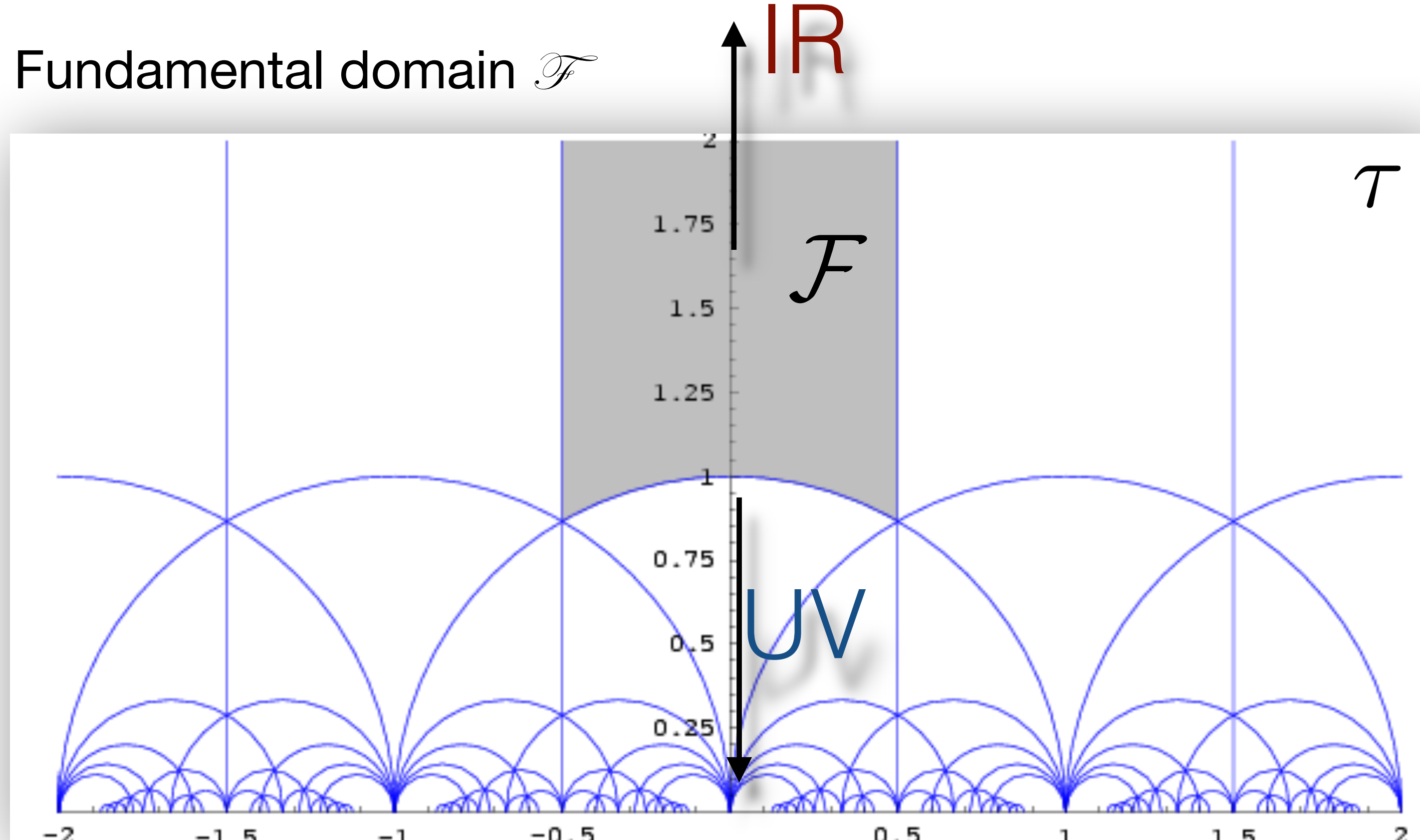
Fundamental domain $\mathscr{F}$



$$\tau = \tau_1 + i\tau_2$$

$$\left(\mathcal{M}^2 = \frac{1}{4\pi^2\alpha'} = \frac{M_s^2}{4\pi^2} \right)$$

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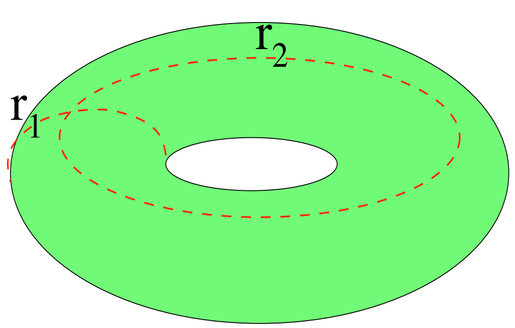


$$\Lambda \equiv -\tfrac{1}{2}\mathcal{M}^D \int_{\mathcal{F}} \frac{d^2\tau}{{\tau_2}^2} Z(\tau) \qquad q = e^{2\pi i \tau}$$

$$= -\frac{1}{2}\mathcal{M}^D \int_{\mathcal{F}} \frac{d^2\tau}{{\tau_2}^{\frac{D}{2}+1}} \sum_{m,n} a_{mn} \overline{q}^m q^n$$

Counts physical (level matched) states weighted by statistics at each level

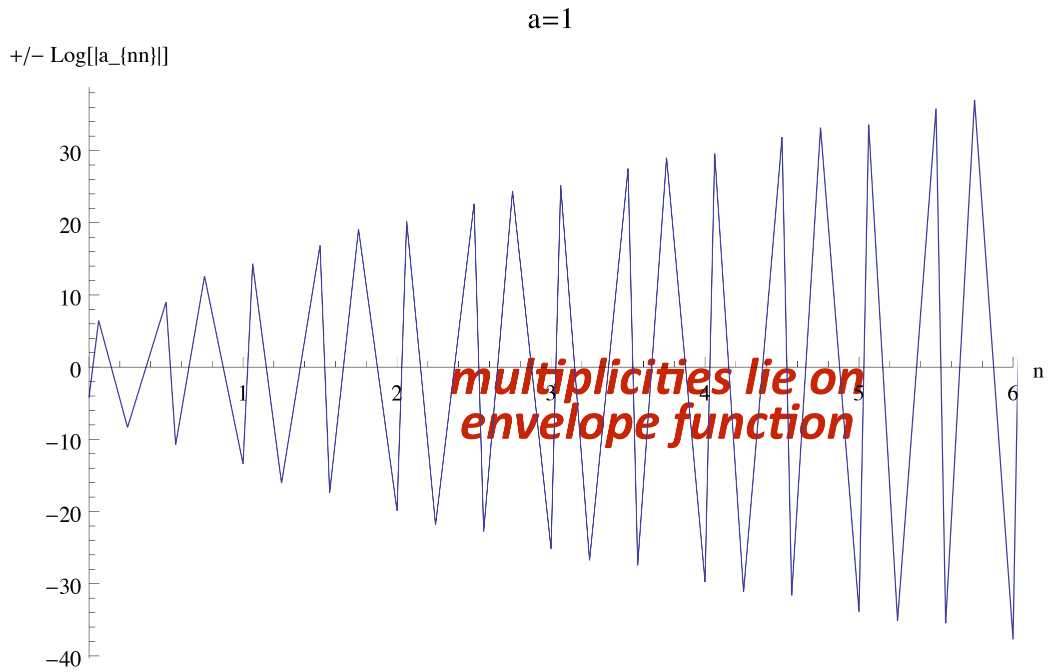
$$\approx -\frac{1}{2}\mathcal{M}^D \int_{M_{UV}^{-2}}^{\mu_{IR}^{-2}} \frac{d\tau_2}{{\tau_2}^{\frac{D}{2}+1}} \sum_n a_{nn}\, e^{-\pi\tau_2\alpha' M_n^2}$$

Due to modular invariance: there's an important way to understand this as a supertrace relation over the infinite tower of physical states. Much more natural and general for what we want to do. Superficially even looks similar to the field theory:

$$\Lambda \quad = \frac{1}{24}\mathcal{M}^2 \mathop{\mathrm{STr}}_{\mathrm{phys}} M^2$$

- Dienes, Misaligned SUSY, 1994
- Dienes, Moshe, Myers 1995

But note this definitely is *not* a field theory object — this supertrace is over the *infinite* string tower of states!!



- This crazy spectrum has finite Λ

Derivation: the integral we need to do in 4D is:

$$\Lambda \quad = -\frac{\mathcal{M}^4}{2}\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2}\,\mathcal{Z}(\tau)$$

Want to write this in terms of physical (level-matched) states whose nett spectral density is:

$$\begin{aligned} g(\tau_2) &= -\frac{\mathcal{M}^4}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1 \mathcal{Z}(\tau) \\ &= -\frac{\mathcal{M}^4}{2}\tau_2^{-1}\mathrm{Str} e^{-\pi\tau_2\alpha' M^2} \end{aligned}$$

Rankin-Selberg: unfold integral to the “critical strip” by convoluting it with an Eisenstein series:

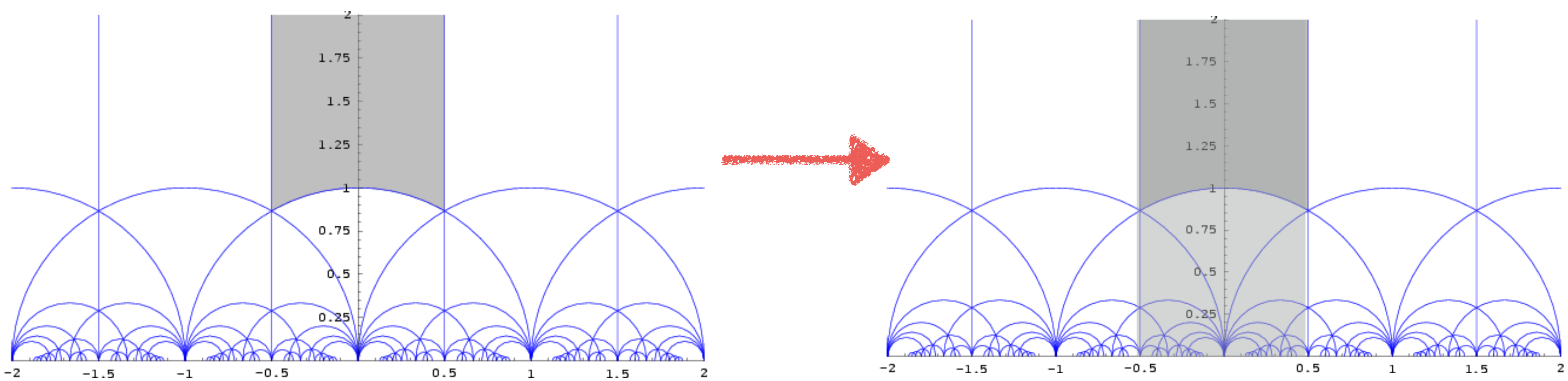
- Rankin, (1940), Selberg (1940), Zagier (1981)
- Angelantonj, Cardella, Elitzur, and Rabinovici
- Angelantonj, Florakis, and Pioline

$$\Lambda \quad = \quad 2\,\mathrm{Res}_{s=1}\left(\mathcal{R}^\star(F,s)\right)$$

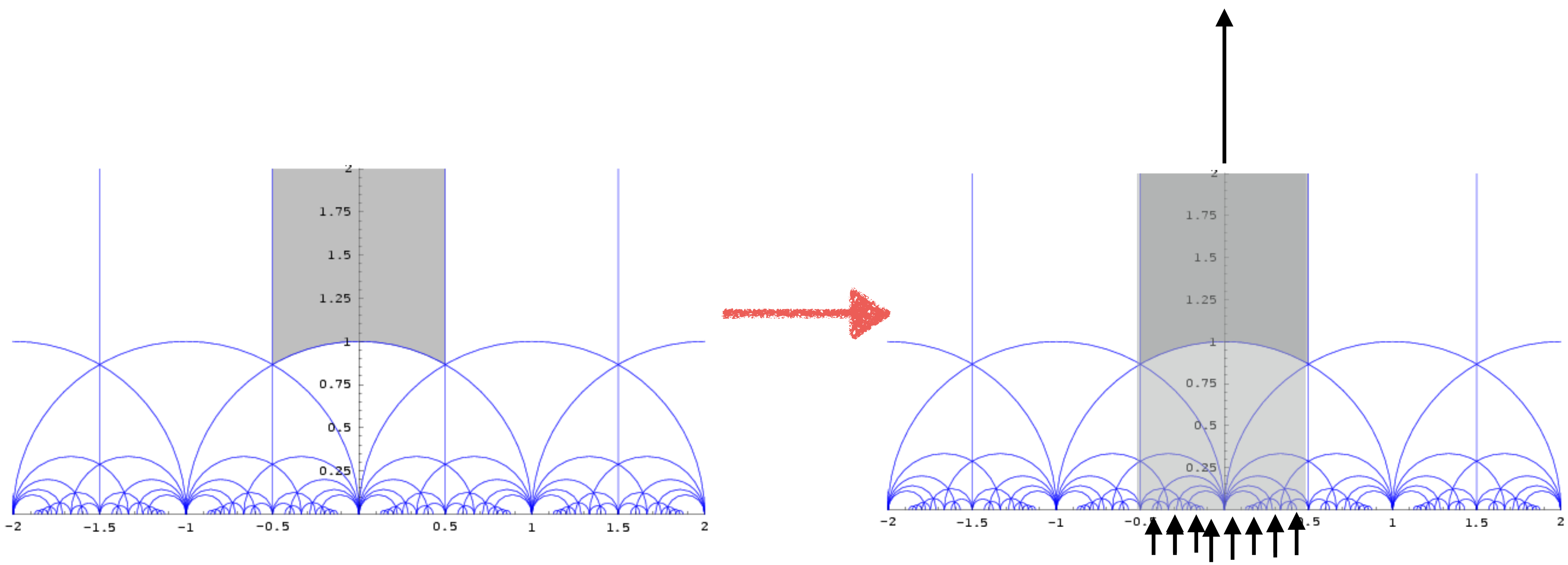
where $\mathcal{R}^\star$ is the Rankin-Selberg transform:

$$\begin{aligned}\mathcal{R}^\star(F,s) \;&=\; \int_0^\infty \frac{d^2\tau}{\tau_2^2}\tau_2^s\pi^{-s}\Gamma(s)\zeta(2s)\,g(\tau_2)\\ &=\; -\frac{\mathcal{M}^4}{2\pi^2}\,\Gamma(s)\Gamma(s-2)\,\zeta(2s)\,\underset{\text{phys}}{\mathrm{STr}}(\pi^2\alpha' M^2)^{2-s}\end{aligned}$$

The whole integral here including the projection to physical states is now looking like:



Note the important difference from the usual picture. There is now clearly no single “IR cusp”. All cusps contribute equally to the integral:



All cusps are equivalent under modular transformations, and there is no “ultra UV” anywhere.

3. Modular constraints on the Higgs mass

Or: a connection between the cosmological constant and the Higgs

First assume that the partition function is a function of the higgs. Then begin with the naive expression:

$$m_\phi^2 \equiv \left. \frac{d^2\Lambda(\phi)}{d\phi^2} \right|_{\phi=0}$$

where $\Lambda(\phi) \equiv -\frac{\mathcal{M}^4}{2} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \mathcal{Z}(\tau, \overline{\tau}, \phi)$

$\mathcal{Z}$ has to be a combination of modular functions which can also be written as a lattice sum:

$$\mathcal{Z}(\tau, \overline{\tau}, \phi) = \tau_2^{-1} \frac{1}{\overline{\eta}^{12} \eta^{24}} \sum_{\mathbf{Q}_L, \mathbf{Q}_R} (-1)^F \overline{q}^{\mathbf{Q}_R^2/2} q^{\mathbf{Q}_L^2/2}$$

So naively we need to do a modular integral of the following form:

$$\frac{\partial^2 \mathcal{Z}}{\partial\phi^2} = \tau_2^{-1} \frac{1}{\overline{\eta}^{12}\eta^{24}} \sum_{\mathbf{Q}_L, \mathbf{Q}_R \in L} (-1)^F \, X \; \overline{q}^{\mathbf{Q}_R^2/2} q^{\mathbf{Q}_L^2/2}$$

where everything hinges on the summand insertion *X,* coming from the derivatives

$$X \equiv \pi i \frac{\partial^2}{\partial\phi^2}(\tau \mathbf{Q}_L^2 - \overline{\tau}\mathbf{Q}_R^2) - \pi^2 \left[\frac{\partial}{\partial\phi}(\tau \mathbf{Q}_L^2 - \overline{\tau}\mathbf{Q}_R^2) \right]^2$$

This in turn requires us to work out the most general shifts of the following form that maintain modular invariance:

$$\begin{aligned} \mathbf{Q}_L &\to \mathbf{Q}_L + \sqrt{\alpha'}\phi \mathbf{Q}_a + \frac{1}{2}\alpha'\phi^2 \mathbf{Q}_b + \ldots \\ \mathbf{Q}_R &\to \mathbf{Q}_R + \sqrt{\alpha'}\phi \tilde{\mathbf{Q}}_a + \frac{1}{2}\alpha'\phi^2 \tilde{\mathbf{Q}}_b + \ldots , \end{aligned}$$

Turns out the allowed charge shifts can be decomposed as gauge generators acting on $\mathbf{Q} = \{\mathbf{Q}_L\,, \mathbf{Q}_R\}$ that take the following form:

$$\mathcal{T} \sim \left(\begin{array}{cccccc|cc} & & & & & \mathbf{t} & 0 & \tilde{\mathbf{t}} \\ & & & & & \mathbf{0} & 0 & \tilde{\mathbf{0}} \\ & & \mathbf{0}_{5\times 5} & & & \mathbf{0} & 0 & \tilde{\mathbf{0}} \\ & & & & & \mathbf{0} & 0 & \tilde{\mathbf{0}} \\ & & & & & \mathbf{0} & 0 & \tilde{\mathbf{0}} \\ -\mathbf{t}^t & \mathbf{0}^t & \mathbf{0}^t & \mathbf{0}^t & \mathbf{0}^t & \mathbf{t}_{11} & \mathbf{0}^t & \mathbf{t}_{12} \\ \hline 0 & 0 & 0 & 0 & 0 & \mathbf{0} & 0 & \tilde{\mathbf{0}} \\ -\tilde{\mathbf{t}}^t & \tilde{\mathbf{0}}^t & \tilde{\mathbf{0}}^t & \tilde{\mathbf{0}}^t & \tilde{\mathbf{0}}^t & -\mathbf{t}_{12}^t & \tilde{\mathbf{0}}^t & \mathbf{t}_{22} \end{array}\right)$$

where e.g. $\begin{pmatrix} \mathbf{Q}_a \\ \tilde{\mathbf{Q}}_a \end{pmatrix} = \mathcal{T} \cdot \mathbf{Q}$

and where the *t* are row vectors.

But then at the end of the day the relevant part of the allowed summand *X* is (**almost**) given by

$$X \;=\; -\,\pi\alpha'\tau_2\,\partial_\phi^2 M^2 + \left(\pi\alpha'\tau_2\right)^2 \left(\partial_\phi M^2\right)^2$$

Almost but not quite: the shifts in $\mathbf{Q}$ induced by the Higgs correspond to coordinate shifts of the modular forms (actually the Higgs *is* a linear combination of these coordinates). For the Higgs derivatives to be modular *covariant* we require a modular completion which is found to be universal:

$$X \longrightarrow X + \frac{\xi}{4\pi^2 \mathcal{M}^2} \qquad \xi = -\mathrm{Tr}(\mathcal{T}_{21}\mathcal{T}_{12})$$

Note that this cosmological constant contribution is due to the modular anomaly of the original X. This universal term would in most practical cases be identified as a Higgs dependent shift in the volume of the compactification space (e.g. 10D —> 4D compactification) which is implicit in the Z charge lattice.

So finally putting this all into Rankin-Selberg we get … ta da !

$$m_\phi^2 = \frac{\xi}{4\pi^2}\frac{\Lambda^{(1)}}{\mathcal{M}^2} + \frac{1}{24}\mathcal{M}^2 \mathrm{Str}\partial_\phi^2 M^2 +$$

$$\mathrm{STr}_{M=0}(\partial_\phi M^2)^2 \times \infty + \mathrm{STr}_{M>0}(\partial_\phi M^2)^2 \times 0$$

Wait. What?!

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4. Regularisation and renormalisation

Or: see how it runs!

The quartic terms are precisely those terms that should be logarithmically dependent on RG scale. But we didn't yet put in any physical RG scale! So at the moment they can only return infinity if the state is massless (or zero if it is massive).

Generally need to find a way to regulate the theory at some IR scale μ in order to extract a physical "running"

Typically to do RG in string theory we subtract the logarithmically divergent states (i.e. the massless "cusp" contribution to the partition function).

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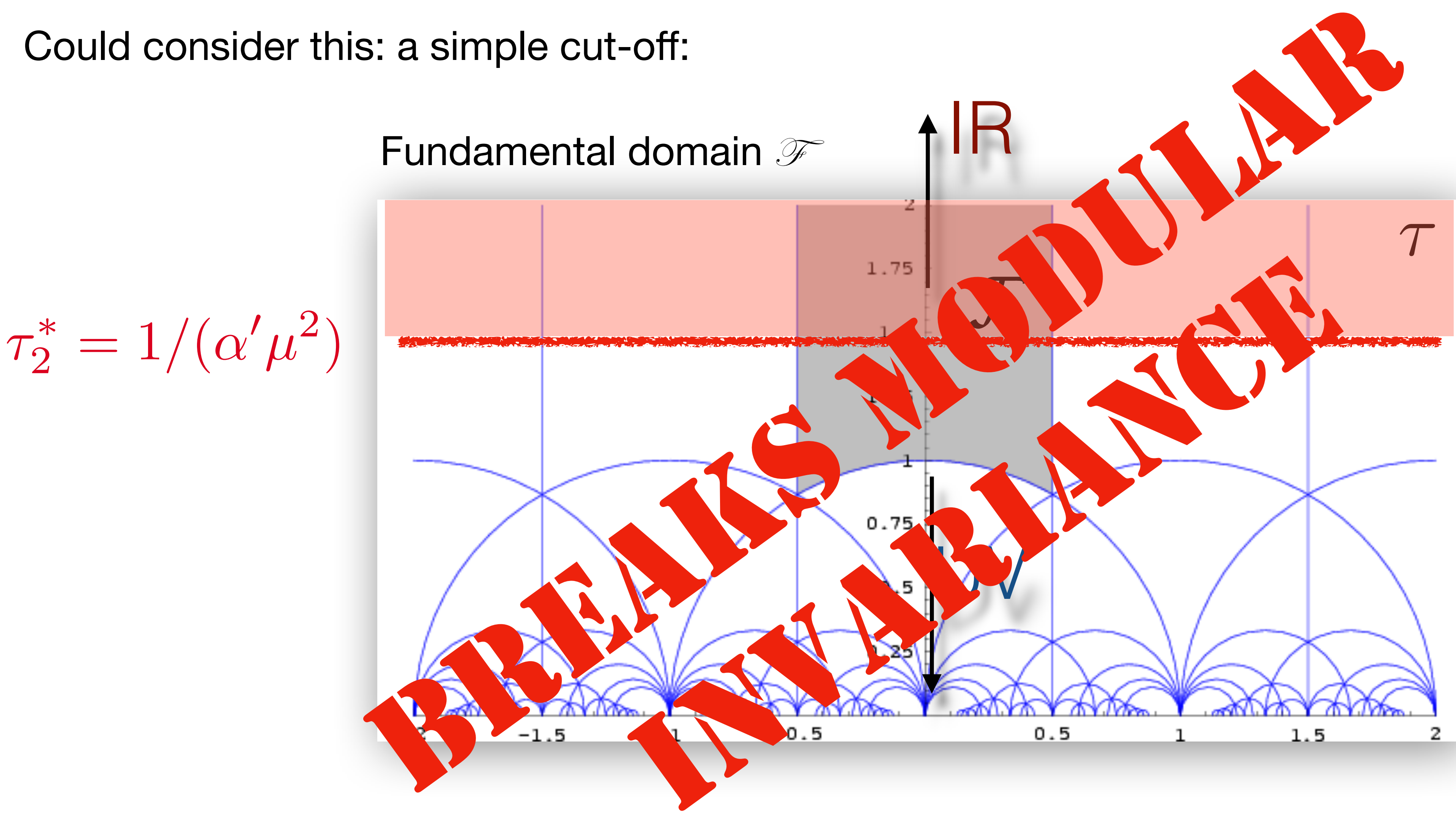
How should we do this? Let's just think about the general modular integral: $I \equiv \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} F(\tau,\overline{\tau})$

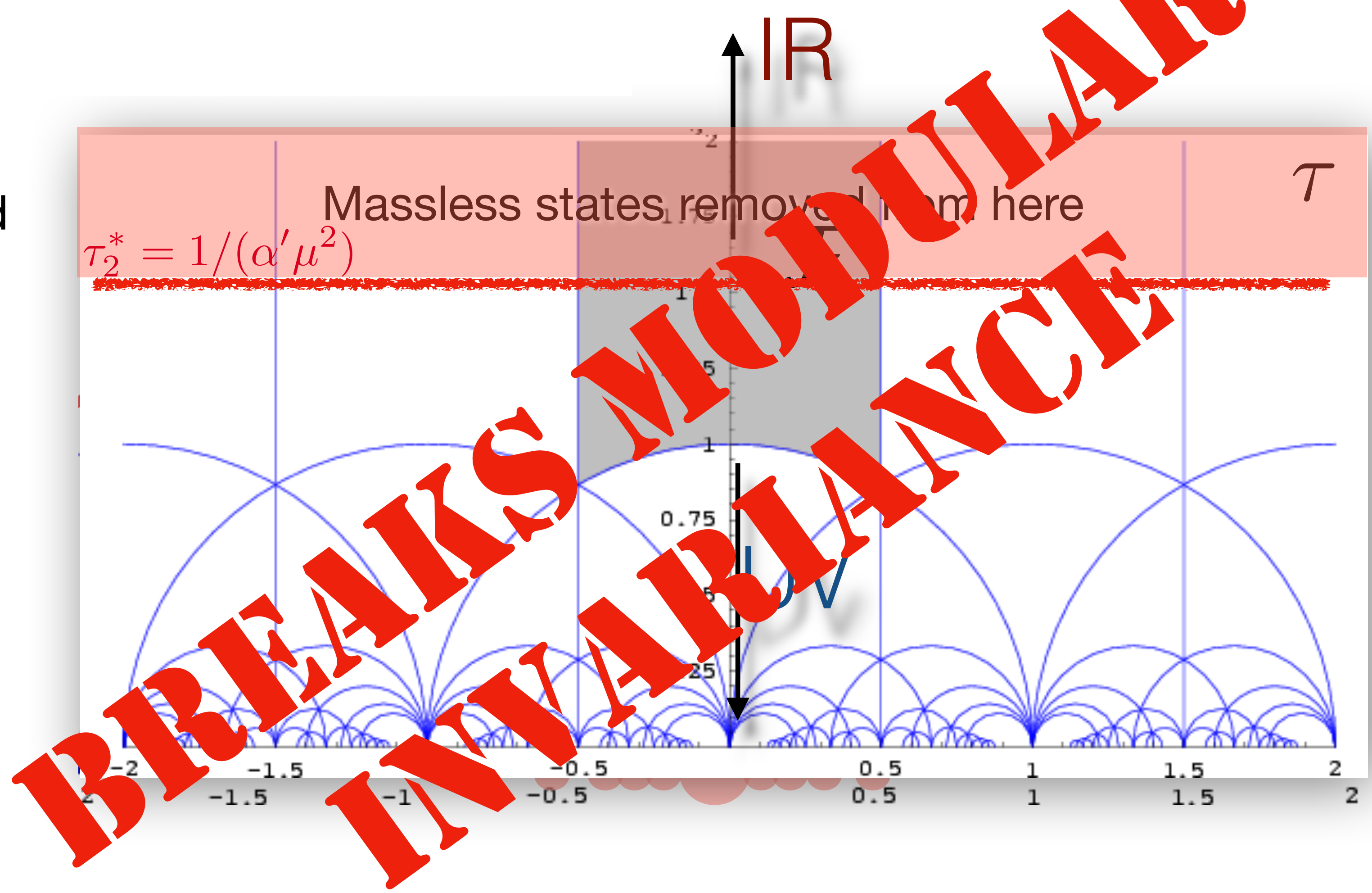
Could consider this: a simple cut-off:



What about modified regulator: subtract just the massless contribution in the IR sector — closest to the traditional RG method (Kaplunovsky)

$$\widehat{I}(\tau_2^*) = \int_{\mathcal{F}_{\tau_2^*}} \frac{d^2\tau}{\tau_2^2} F + \int_{\mathcal{F}-\mathcal{F}_{\tau_2^*}} \frac{d^2\tau}{\tau_2^2} (F - c_1\tau_2)$$

In a tour-de-force in 1981 Zagier showed this can also be written as a Str formula…

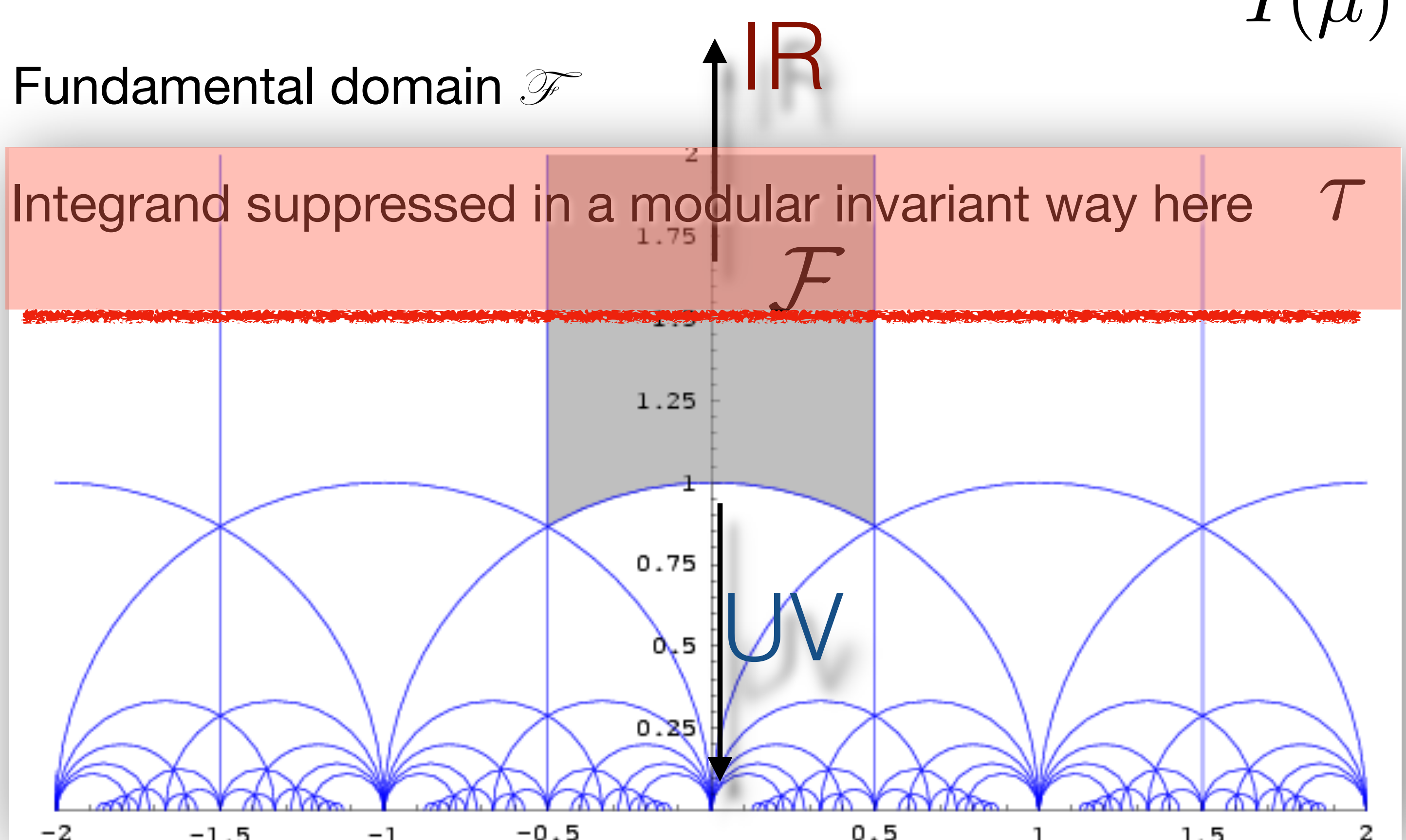


So far, and traditionally, we always think of stringy “threshold corrections” and match them to an effective field theory (EFT). But arguably this approach …

- could never yield a fully modular invariant answer as the EFT is by definition not modular invariant
- cannot give Wilsonian renormalisation: my choice of whether the electron is light enough to be called massless and be subtracted is completely arbitrary and will always break modular invariance

Instead we must abandon the idea of going to an EFT and introduce a modular invariant Wilsonian cut-off instead:

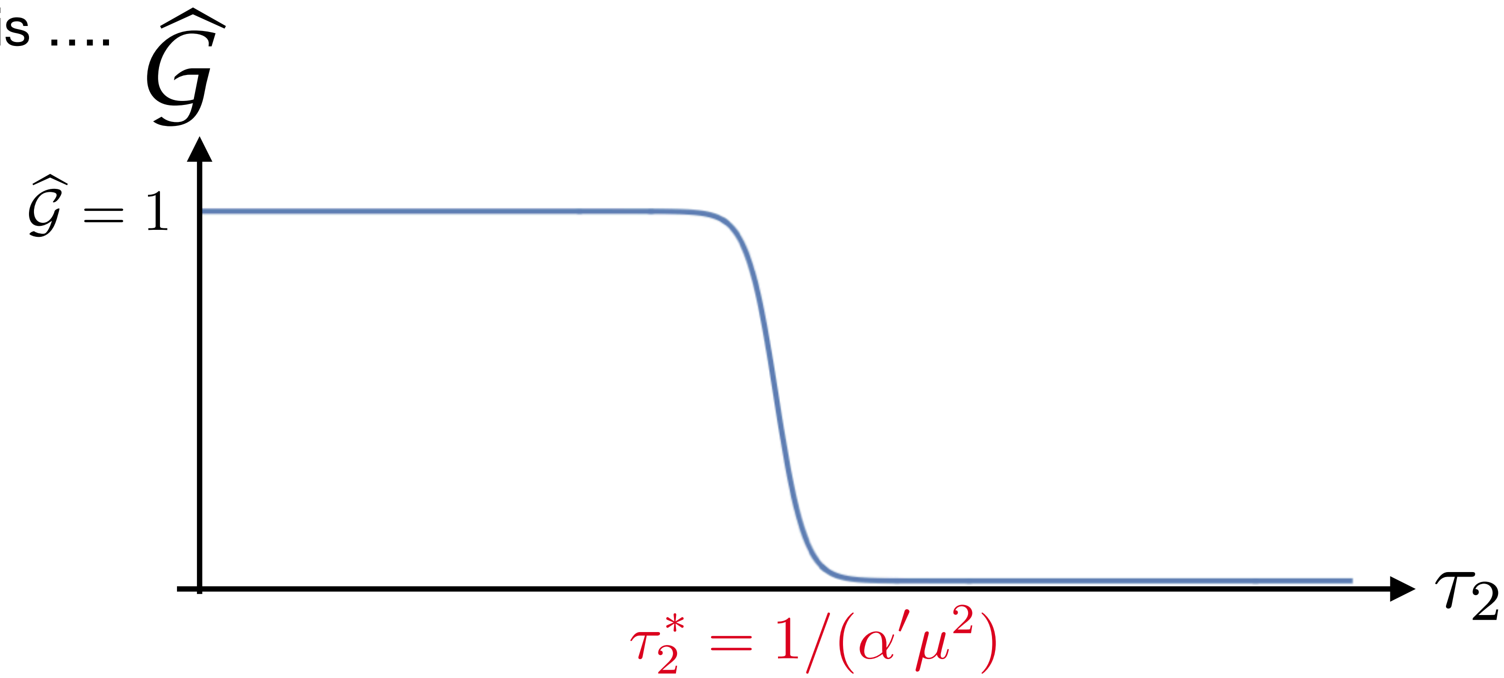
$$\widehat{I}(\mu) = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2}\,\widehat{\mathcal{G}}(\mu,\tau,\overline{\tau})\,F(\tau,\overline{\tau})$$



Required properties of Wilsonian regulator, $\widehat{\mathcal{G}}$:

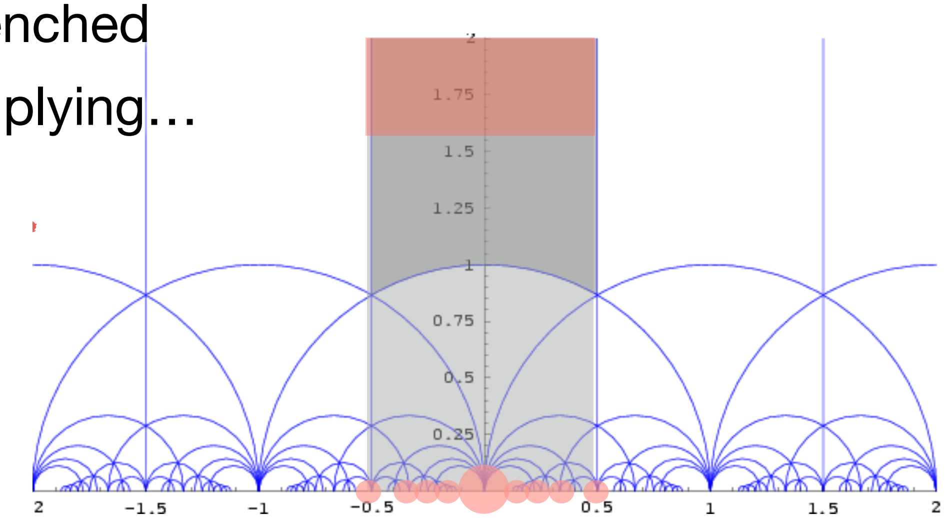
$$\widehat{I}(\mu) = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2}\, \widehat{\mathcal{G}}(\mu,\tau,\overline{\tau})\, F(\tau,\overline{\tau})$$

- a) Is itself a modular function
- b) Should look like this ….



- c) Remember, our goal is to write everything as a supertrace which ultimately means an integral over the critical strip …This only makes sense if all the cusps are quenched equally. In other words: all the cusps are equivalent IR cusps, implying…

$$\tau_2^* \equiv 1/\tau_2^* \implies \widehat{\mathcal{G}}(\mu,\tau,\overline{\tau}) = \widehat{\mathcal{G}}(M_s^2/\mu,\tau,\overline{\tau})$$



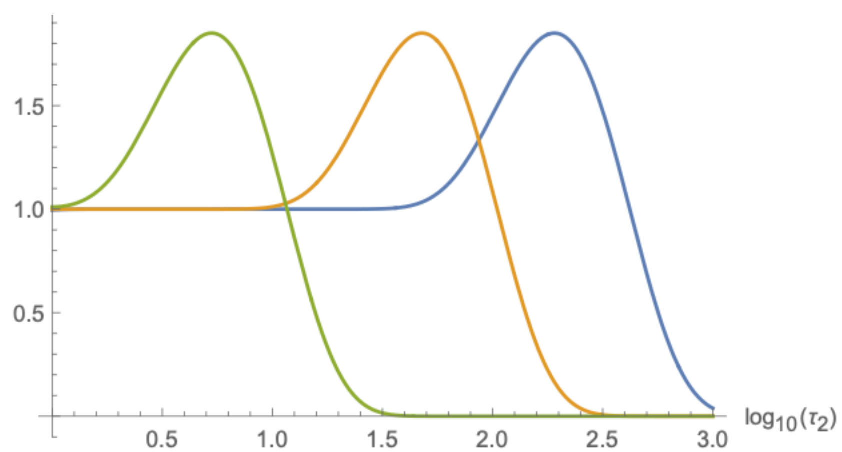
Such a modular invariant regulator very nearly exists (modulo the last property) (Kiritsis, Kounnas)

- Take the circle partition function with radius defined by parameter $a \equiv \sqrt{\alpha'}/R$:

$$Z_{\rm circ}(a,\tau) = \sqrt{\tau_2} \sum_{m,n\in\mathbb{Z}} \overline{q}^{(ma-n/a)^2/4}\, q^{(ma+n/a)^2/4}$$

- Then a suitable cut-off function that obeys all these properties is …

$$\widehat{\mathcal{G}}_\rho(a,\tau) \;=\; \frac{\rho/(\rho-1)}{1+\rho a^2}\; a^2 \frac{\partial}{\partial a}\Big[Z_{\rm circ}(\rho a,\tau) - Z_{\rm circ}(a,\tau)\Big]$$



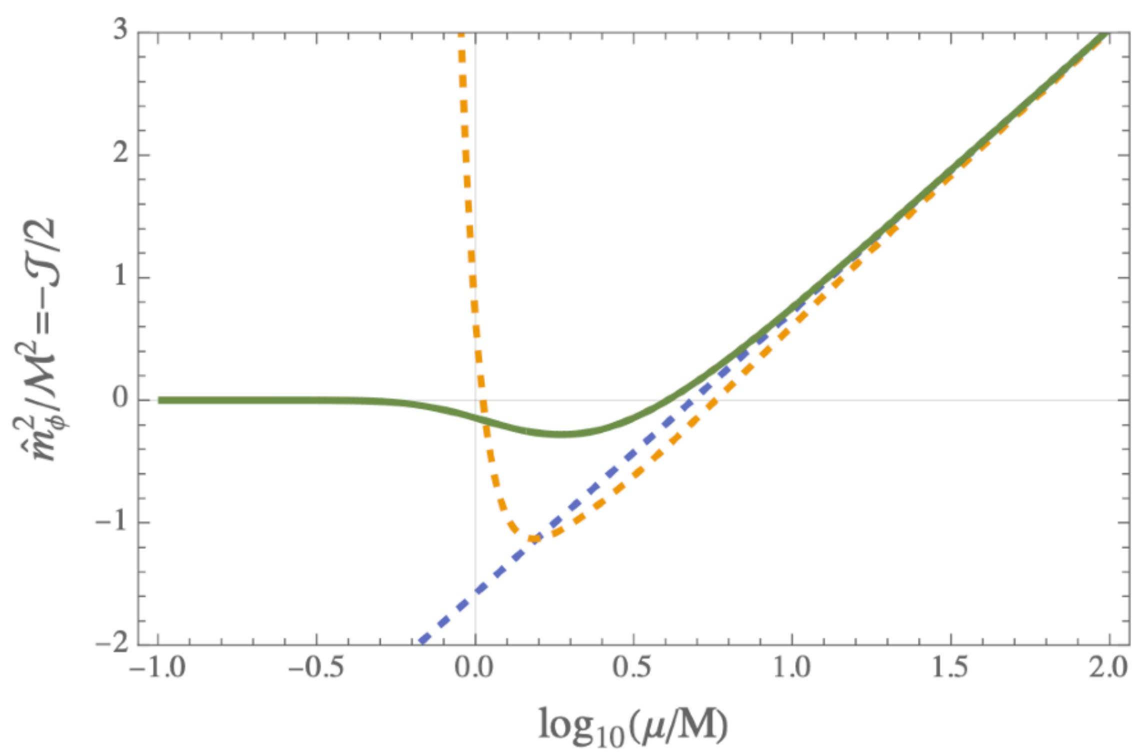
$$\mu^2(\rho,a) \;\equiv\; \frac{\rho a^2}{\alpha'} \quad\Longrightarrow\quad \tau_2^* = \rho a^2$$

- The nice thing about this function is we can find the simpler $P(a) = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2}\, F(\tau)\, Z_{\rm circ}(a,\tau)$ as a supertrace, and then take the derivative

The result is a smooth modular invariant answer with an IR cut-off

Complicated sum of Bessel functions, but has the following asymptotic behaviour …

$$\hat{m}_\phi^2(\mu) = \frac{\xi}{4\pi^2}\frac{\Lambda^{(1)}}{\mathcal{M}^2} + \frac{1}{24}\mathcal{M}^2\mathrm{Str}\partial_\phi^2 M^2 + \frac{1}{64\pi^2}\,\mathrm{Str}_{0<M<\mu}\partial_\phi^2 M^4 \log\left(\frac{e^{2(\gamma+1)}\rho^{\frac{\rho+1}{\rho-1}}}{4}\times\frac{M^2}{\mu^2}\right)$$



Bessel function sum turns over for states once the scale μ drops below M

Below the mass of all states (that couple to the Higgs) there is no further contribution

The result is a sum over all states *as if they had logarithmically run up from their mass*.

It is by construction symmetric around the string scale.

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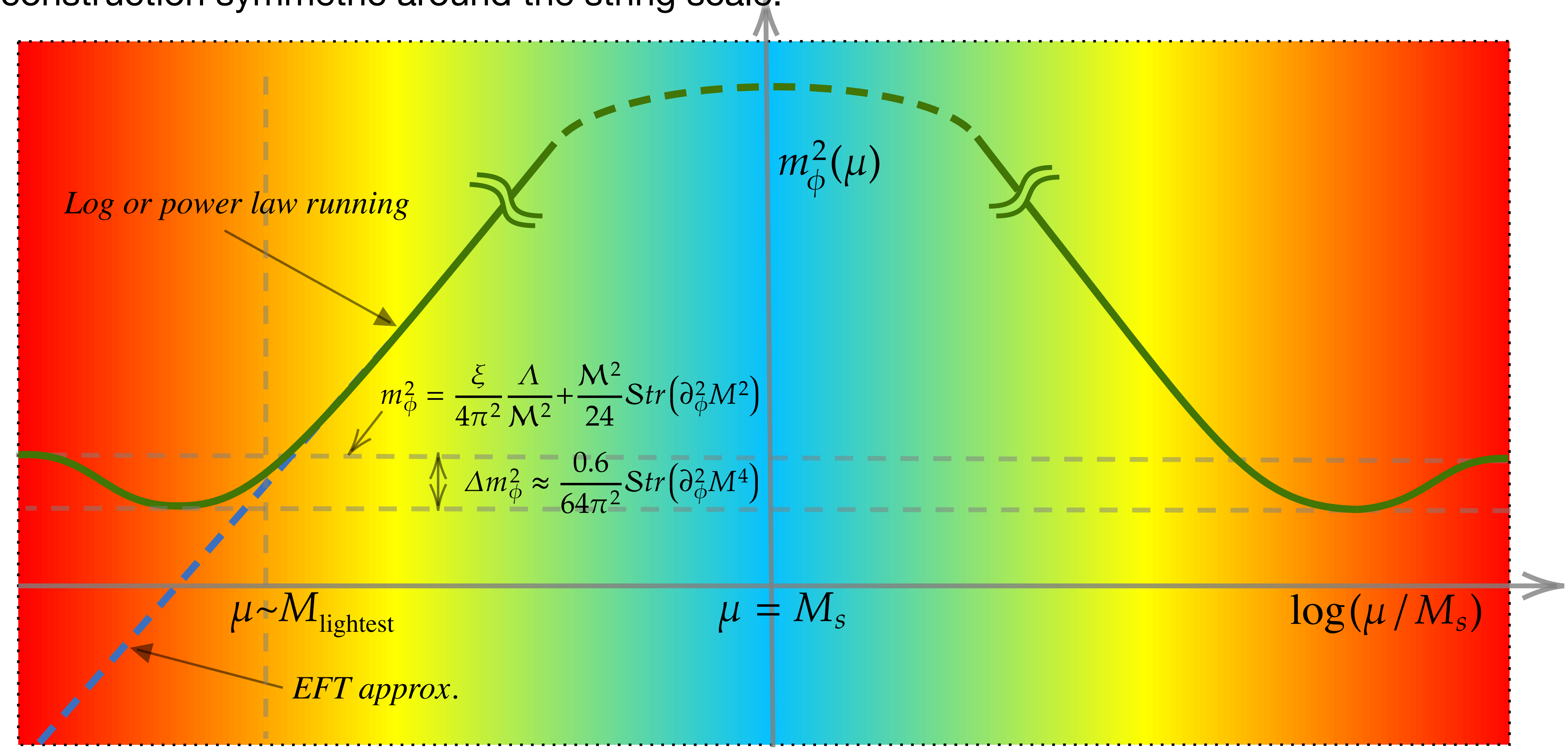

$$\lim_{\mu \to 0} \hat{m}_\phi^2(\mu) = \frac{\xi}{4\pi^2}\frac{\Lambda}{\mathcal{M}^2} + \frac{1}{24}\mathcal{M}^2 \mathrm{Str}\partial_\phi^2 M^2$$

5. Conclusions

- We have developed a general supertrace formula for the Higgs, that plays the role for all generic modular invariant theories that the Coleman-Weinberg potential plays for field theory.
- A modular invariant regulator provides a natural Wilsonian cut-off and definition of RG scale. Gives meaning where the EFT fails, and retains the predictivity of the UV complete theory.
- Operators such as the Higgs mass can be thought of as “running” to its cusp value: this is both the UV and IR asymptote.
- The Weak/Planck and cosmological constant hierarchy problems are connected in this one operator.
- A single stringy naturalness condition: $\boxed{\mathrm{Str}\,\partial_\phi^2 M^2 \lesssim \frac{24}{\mathcal{M}^2} M_W^2}$