Temperature dependence of the axion mass: a case for master-field simulations

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Based on work done with Leonardo Giusti

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Physics of the axial anomaly

The flavour-singlet axial current satisfies

Adler '69

Bell & Jackiw '69

$$\partial_\mu j^5_\mu(x) = 2N_\mathrm{f}\, q(x) + \text{mass terms}$$

$$q(x) \propto \epsilon_{\mu\nu\rho\sigma} F^a_{\mu\nu}(x) F^a_{\rho\sigma}(x), \qquad Q \equiv \int_x q(x) \in \mathbb{Z}$$ (if A_μ is smooth)

- The η' is not a Goldstone boson

Witten '79

Veneziano '79

$$m_\pi^2 \ll m_{\eta'}^2 \simeq \frac{2N_\mathrm{f}}{F_\pi^2}\, \chi_t\big|_{N_\mathrm{f}=0}$$

$$\chi_t = \frac{\langle Q^2 \rangle}{V} = \int_x \langle q(x) q(0) \rangle$$ ("topological susceptibility")

- Strong CP problem: Why is there no θQ term in QCD?

Peccei–Quinn & the axion

Promote flavour-singlet axial U(1) to a non-anomalous chiral symmetry

$$S_{\rm axion} = \int_x \left\{ \frac{1}{2}\partial_\mu a(x)\partial_\mu a(x) + \frac{i}{M} a(x) q(x) \right\}$$

Peccei & Quinn '77

- ⋆ Axial U(1) transformation: $a(x) \to a(x) + \omega$
- ⋆ $\langle a(x) \rangle = 0$ minimizes effective action $\Rightarrow$ CP is naturally preserved
- ⋆ Axion = Goldstone boson with mass $m_a^2 = \dfrac{\chi_t}{M^2} + \ldots$

The axion is a dark-matter candidate

$\Rightarrow$ *need to understand the temperature dependence of its mass at* $T > T_{\rm c}$

Is χ_t a well-defined quantity?

The 2-point function

$$\langle q(x)q(0)\rangle \underset{x\to 0}{\sim} (x^2)^{-4}$$

has a non-integrable short-distance singularity

$\Rightarrow$ its Fourier transform

$$\langle q(x)q(0)\rangle \underset{\mathrm{F.T.}}{=} c_0\Lambda^4 + c_1\Lambda^2 p^2 + c_2(p^2)^2 + (p^2)^3 \int_0^\infty \mathrm{d}s\, \frac{\rho(s)}{s^3(s+p^2)}$$

is determined only up to contact terms

$\sim$*30 years of theoretical development were required to fully resolve this issue!*

Lattice QCD with exact chiral symmetry

$\downarrow$

Representation of χ_t through density chains

$$\int_{x,y,\ldots} \langle P_{rs}(x) S_{st}(y) \ldots S_{uv}(z) \rangle$$

$\downarrow$

and via the Yang–Mills gradient flow

$$\partial_s B_\mu = D_\nu G_{\nu\mu}, \quad B_\mu|_{s=0} = A_\mu$$

$$\chi_t = \int_x \langle q(x,s) q(0,s) \rangle$$

Ginsparg & Wilson '82
Kaplan '92
Hasenfratz et al. '98
Neuberger '98
ML '98

Giust et al. 02
Giusti, Rossi & Testa '04
ML '04

Atiyah & Bott '82
ML '10
ML & Weisz '11
Cè et al. '15

χ_t at temperatures $T \gg T_c$

Common lore says that

thermal masses in QCD $\propto T$ but $\chi_t \propto T^{-7-N_f/3}$

i.e. the axion rapidly becomes very light at high T!

Based on the classical instanton bound

$$S \geq 8\pi^2|Q|,$$

the renormalization group and some ad hoc assumptions

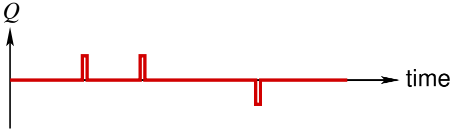
At $T > 0$ the semi-classical approximation is however extremely complicated and its applicability is anyway questionable

Kraan & van Baal '98

How about computing the T-dependence of χ_t in LQCD?

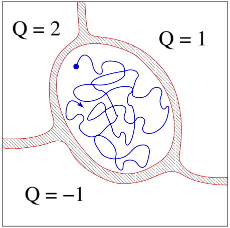
In principle straightforward, but

- Require long runs if $\langle Q^2 \rangle = \chi_t V \ll 1$



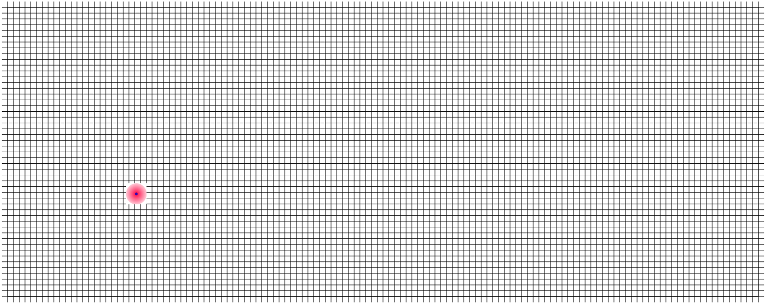
- Lattice spacing a must satisfy $1/a > T$

- Topology-freezing issue



Master fields

Field variables at distances $\gg m_\pi^{-1}$ are statistically independent



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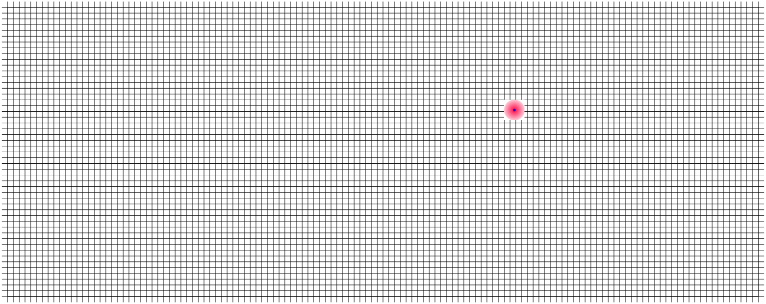

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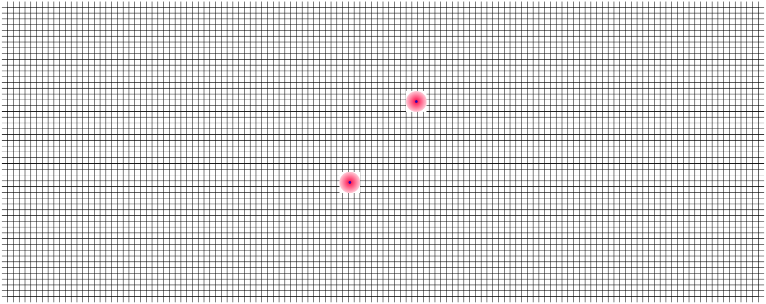

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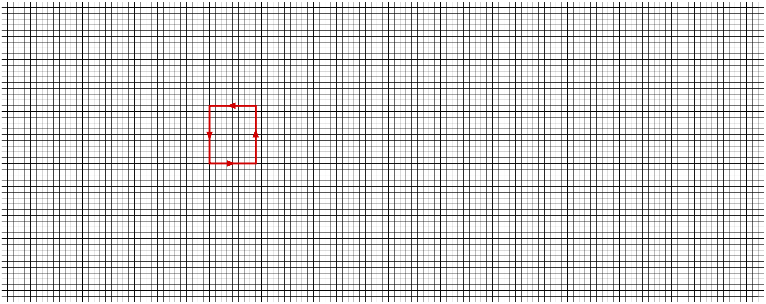

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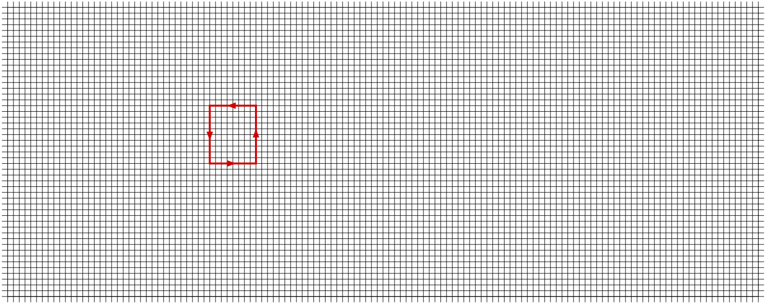


$\Rightarrow$ On an infinite lattice, the expectation values $\langle \mathcal{O} \rangle$ may conceivably be obtained from a single field!

Free-field example

Draw a Gaussian random number $\eta(x)$ at each point x of an infinite lattice

$$\phi(x) = \sum_y K(x-y)\eta(y), \qquad K: \text{ kernel of } (-\Delta + m^2)^{-1/2}$$

$$\Rightarrow \quad \lim_{V\to\infty} \frac{1}{V} \sum_{z\in V} \phi(x_1+z)\ldots\phi(x_n+z) = \langle \phi(x_1)\ldots\phi(x_n)\rangle$$

- The way the subvolume V grows is irrelevant
- Master fields are not unique!

Translation averages

Let $\mathcal{O}(x)$ be some quasi-local observable with footprint $\ll V$

$\Rightarrow$ The translation average

$$\langle\!\langle \mathcal{O} \rangle\!\rangle = \frac{1}{V} \sum_x \mathcal{O}(x)$$

has mean $\langle \mathcal{O} \rangle$ and variance

$$\frac{1}{V} \sum_x \langle \mathcal{O}(x)\mathcal{O}(0) \rangle_{\mathrm{c}}$$

Moreover, its distribution is Gaussian up to higher-order corrections in $1/V$

Master-field simulations

ML '17

- Consider a physically large lattice with periodic bc
- Run a simulation as usual, stop after thermalization and use the last configuration
- $\langle \mathcal{O} \rangle = \langle\langle \mathcal{O} \rangle\rangle$ up to statistical errors of $\mathrm{O}(V^{-1/2})$

Cost balance

$$\left.\begin{array}{l}\text{large lattice}\\ 1\ \text{configuration}\end{array}\right\} \simeq \left\{\begin{array}{l}\text{small lattice}\\ N\ \text{configurations},\ N = \dfrac{V_{\mathrm{large}}}{V_{\mathrm{small}}}\end{array}\right.$$

E.g. $L = 30\,\mathrm{fm}$ instead of 10^4 configurations at $L = 3\,\mathrm{fm}$

Statistical error estimation

$$\frac{1}{V}\sum_{x}\langle\mathcal{O}(x)\mathcal{O}(0)\rangle_{\mathrm{c}} = \frac{1}{V}\Bigg\{\sum_{|x|\leq R}\langle\mathcal{O}(x)\mathcal{O}(0)\rangle_{\mathrm{c}} + \mathrm{O}(\mathrm{e}^{-mR})\Bigg\}$$

$$= \frac{1}{V}\Bigg\{\sum_{|x|\leq R}\langle\!\langle\mathcal{O}(x)\mathcal{O}(0)\rangle\!\rangle_{\mathrm{c}} + \mathrm{O}(\mathrm{e}^{-mR}) + \mathrm{O}(V^{-1/2})\Bigg\}$$

Note: Must have both $mR \gg 1$ and $L \gg R$

Topology freezing?

Becomes irrelevant at large V, since

$$\langle\mathcal{O}\rangle_{Q=\nu} = \langle\mathcal{O}\rangle + (r^2-1)\frac{\langle\mathcal{O}Q^2\rangle_{\rm c}}{2\chi_t V} + {\rm O}(V^{-2}), \qquad r \equiv \frac{\nu}{\sqrt{\langle Q^2\rangle}}$$

for any quasi-local scalar observable $\mathcal{O}$

Brower et al. '03, Aoki et al. '07

Note: Must have $\chi_t V \gg 1$

$\Rightarrow$ *Master-field simulations can bypass both the rare-events and the topology-freezing issue!*

Master-field simulations of the SU(3) gauge theory

Giusti & ML '18

Lattice	a [fm]	T [MeV]	L [fm]
6×256^3	0.073	449	19
8×384^3	0.055	449	21
12×512^3	0.037	449	19
6×512^3	0.055	596	28
8×768^3	0.041	596	32
12×1024^3	0.028	596	28

Additionally simulated lattices up to 256^4 for accurate scale-setting through the reference gradient-flow time t_0

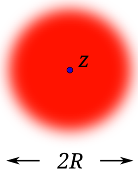
Computation of χ_t

For given gradient-flow time s and radius R, estimate

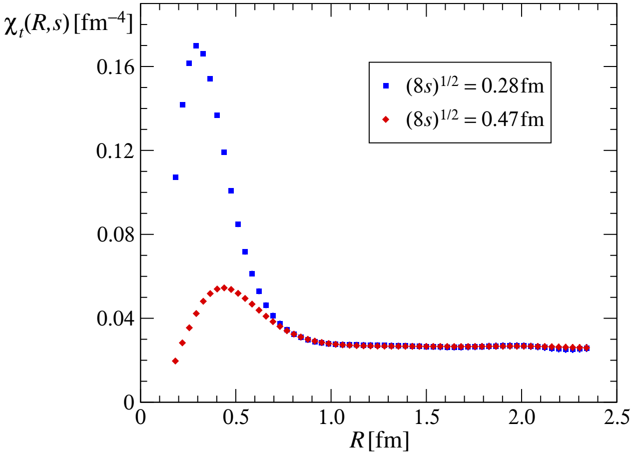
$$\chi_t(R,s) = a^4 \sum_{x_0} \sum_{|\vec{x}|\leq R} \langle q(x,s) q(0,s) \rangle$$

through the translation averages of the observable

$$\mathcal{O}(z) = a^4 \sum_{x_0} \sum_{|\vec{x}|\leq R} q(x+z,s) q(z,s)$$



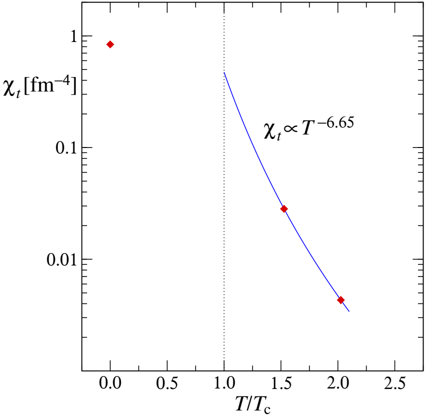
Example: $T = 1.5\,T_\mathrm{c},\ L = 19\,\mathrm{fm}$



Results in the continuum limit

T/T_c	T [MeV]	χ_t [fm^{-4}]
0.0	0	0.839(9)
1.5	449	0.0283(15)
2.0	596	0.00431(34)

- Typically $\chi_t V = \mathrm{O}(100)$
- Fixed-topology effects $< 1\%$



Conclusions

- ⋆ *Rapid decay of $m_a(T)$ confirmed with unprecedented conceptual clarity and precision*
- ⋆ *First physics application of master-field simulations!*
- ⋆ *The nearly perfect cancellation of short- and long-distance contributions to χ_t however remains a bit of a mystery*

Conclusions (cont.)

Master-field simulations of QCD may have many uses

- Computations of spectral densities and total hadronic decay widths

  Hansen, Meyer & Robaina '17
  Bulava & Hansen '19
  Hansen, Lupo & Tantalo '19

- Thermodynamic quantities below and above T_c

First master-field simulations of 2+1-flavour QCD at $T = 0$ are now under way

Bruno et al. (Bern-CERN-Edinburgh-Odense-Plymouth)